Close Thu: 14.4, 14.7(1)
Close Tue: 14.7(2),15.1
Close Next Thu: 15.2, 15.3 (integrating!)
Office Hours Today: 1:30-3:00pm (Smith 309)
Math Tutors: 9:30am–9:30pm (Com B-014)
CLUE Tutors: 7:00pm – midnight (Mary Gates)

Entry Task: Find the tangent plane, the linear approximation and total differential for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at (x,y) = (2,1).

14.4/14.7 Tangent Planes & Max/Min

Tangent Plane Summary & Applications

Tangent Plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximation:

$$L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Total Differential:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

Concepts:
If $(x, y) \approx (x_0, y_0)$, then
$f(x, y) \approx L(x, y)$ and $\Delta z \approx dz$.

Applications

1. *Linear Approximation*:
   "Near" the points (x_0,y_0) the tangent plane and surface z-values are close.

2. *Differentials*: Same idea in terms of differences.
   Actual changes on surface are:
   $\Delta x=x-x_0$, $\Delta y=y-y_0$, $\Delta z=f(x,y) - f(x_0,y_0)$
   Approx changes on tangent plane:
   $dx= x - x_0, dy = y - y_0$, $dz = z - z_0$

Example:
Use the linear approximation of

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$ to approximate the value of $f(2.1,0.9)$.

14.7 Local Max/Min

Consider a surface z = f(x,y).

Some Terminology:

A **local maximum** occurs at (a,b) if f(a,b) is larger than *all* values "near" it (top of a hill).

A **local minimum** occurs at (a,b) if f(a,b) is smaller than *all* values "near" it (bottom of a valley).

A **critical point** is a point (a,b) where **BOTH**

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

or where either partial doesn't exist.

Note: If a local max/min occurs at a point, then that point must be a critical point!

Example:

Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

Second Derivative Test

Let (a,b) be a critical point.

Find all **second** partials at (a,b)

$(f_{xx}(a,b), f_{yy}(a,b), f_{xy}(a,b))$

and compute

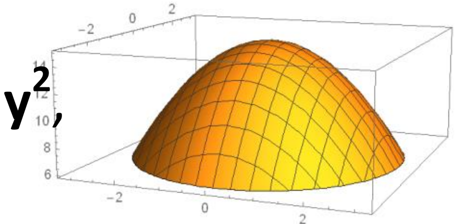
$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

1. If **D > 0**, then the concavity is the same in all directions. So
   (a) If $\mathbf{f_{xx} > 0}$, then it is concave up in all directions. **Local minimum.**
   (b) If $\mathbf{f_{xx} < 0}$, then it is concave down in all directions. **Local maximum.**

2. If **D < 0**, then the concavity changes in some direction.
   We say (a,b) is a **saddle point**.

3. If **D = 0**, the test is **inconclusive** (need a contour map)

Quick Examples:

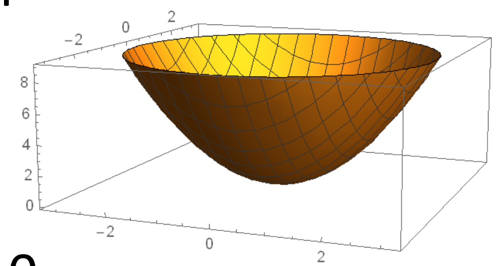


1. $\mathbf{f(x,y) = 15 - x^2 - y^2}$,

Critical pt: (0,0).

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (-2)(-2)-(0)^2 = 4$

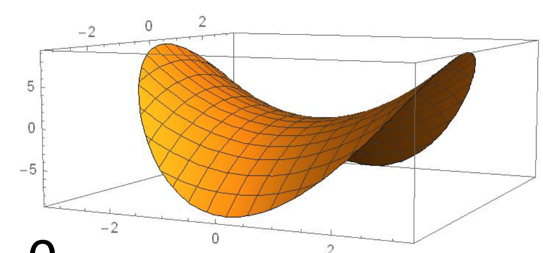


2. $\mathbf{f(x,y) = x^2 + y^2}$,

Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,

$D = (2)(2)-(0)^2 = 4$



3. $\mathbf{f(x,y) = x^2 - y^2}$

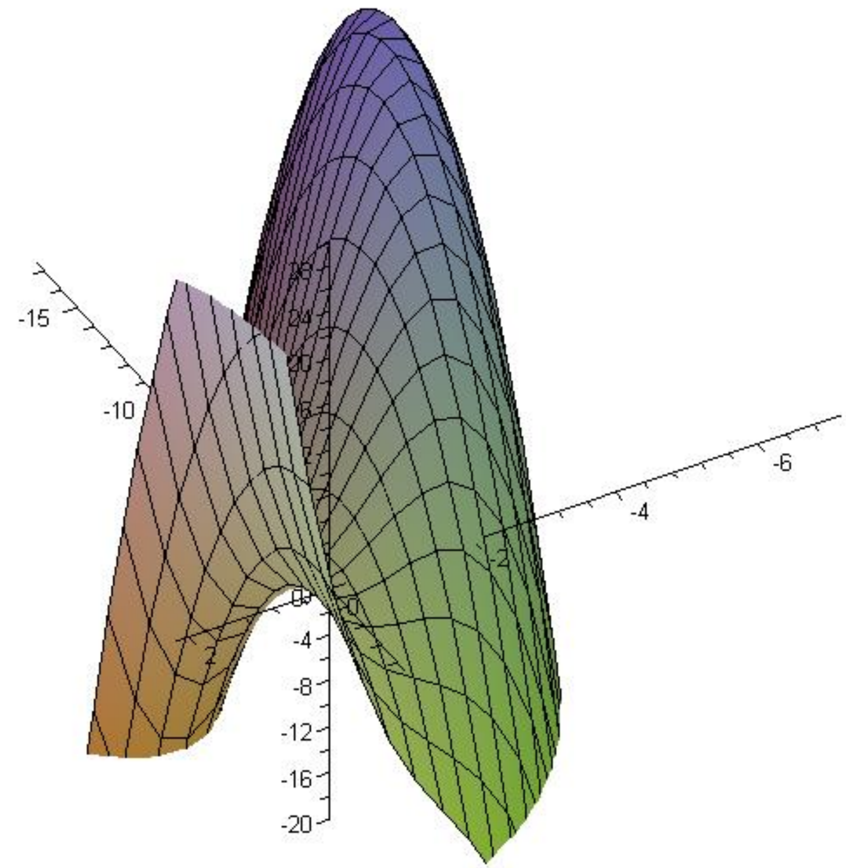
Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (2)(-2)-(0)^2 = -4$

Example: Find and classify all critical points for

$$f(x, y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$



Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

3. Find and classify all critical points for

$$f(x,y) = x^2y - 9y - xy^2 + y^3$$

Global Max/Min

Consider a surface f(x,y) over a particular region R on the xy-plane.

An **absolute/global maximum** over R is the largest z-value over R.

An **absolute/global minimum** over R is the smallest z-value over R.

Key fact (Extreme value theorem)
The absolute max/min must occur at either

1. A critical point, or
2. A boundary point.

Example: Let R be the triangular region in the xy-plane with corners at (0,-1), (0,1), and (2,-1). Over R, find the absolute max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$

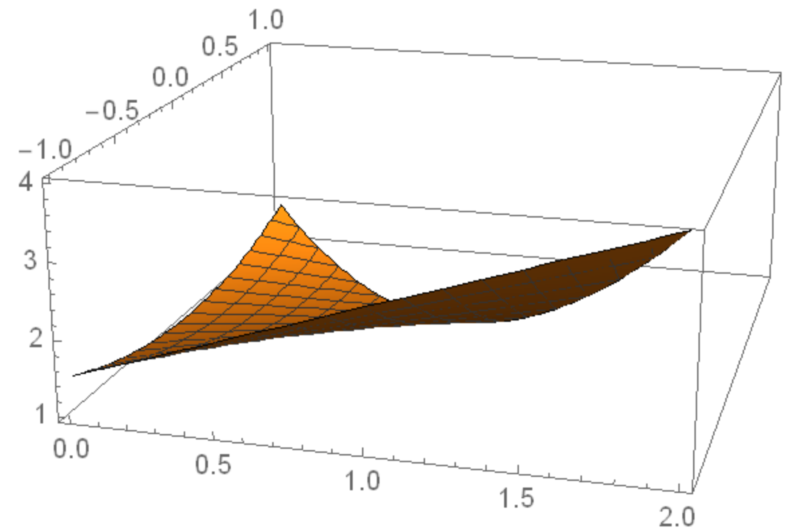
Step 1: Critical points inside region.

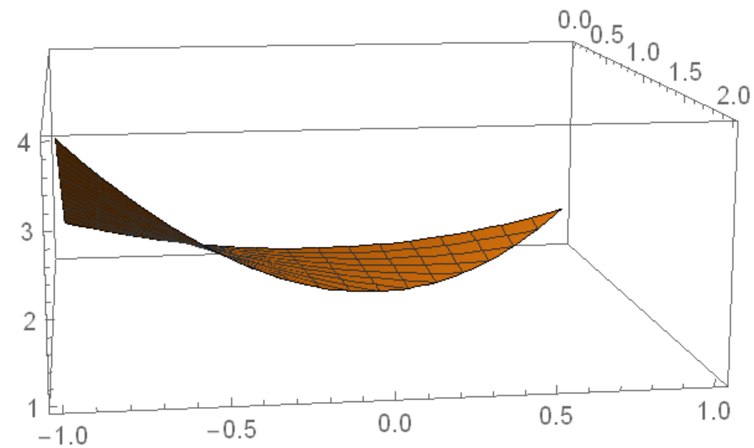
Step 2: Boundaries (the triangle has 3).

i) For each boundary, give equation in terms of x and y.
ii) Find intersection with surface (i.e. substitute).
iii) Find critical numbers and endpoints for this one variable function. Label “corners” (these are the one variable endpoints)

Step 3: Evaluate the function at all points you found in steps 1 and 2.

Biggest output = global max
Smallest output = global min





Homework hints:

In applied optimization problems,

(a) Identify what you are optimizing (objective)
(b) Label Everything.
(c) Identify any given facts (constraints)
(d) Use the constraints and labels to give a 2 variable function for the objective.

HW Examples:

1. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to (4,2,0).

Objective: Minimize **distance** from (x,y,z) points on the cone to the point (4,2,0) given that $z^2 = x^2 + y^2$.

2. Find the dimensions of the box with volume 1000 cm^3 that has minimum surface area.

Objective: Minimize **surface area** given that volume is 1000.